

Off-Forward Parton Distributions in 1+1 Dimensional QCD

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We use two-dimensional QCD as a toy laboratory to study off-forward parton distributions (OFPDs) in a covariant field theory. Exact expressions (to leading order in $1/N_C$) are presented for OFPDs in this model and are evaluated for some specific numerical examples. Special emphasis is put on comparing the $x > \zeta$ and $x < \zeta$ regimes as well as on analyzing the implications for the light-cone description of form factors.

I. INTRODUCTION

Deeply virtual Compton scattering (DVCS) provides a novel tool to explore hadron structure. In contrast to deep-inelastic scattering (DIS), where one measures the imaginary part of the forward Compton amplitude only, DVCS allows measuring the off-forward Compton amplitude. From the parton point of view this implies that DVCS allows measuring off-forward matrix elements of parton correlation functions, i.e. on can access light-cone correlation functions of the form

$$f_\zeta(x, t) \equiv \int \frac{dx^-}{4\pi} \langle p' | \bar{\psi}(0) \gamma^+ \psi(x^-) | p \rangle e^{ixp^+ x^-}, \quad (1.1)$$

where $x^\pm = x^0 \pm x^3$ and $p^+ = p^0 + p^3$ refers to the usual light-cone components and $t \equiv q^2 = (p - p')^2$ is the invariant momentum transfer. The “off-forwardness” (or skewedness) in Eq. (1.1) is defined to be $\zeta \equiv \frac{q^+}{p^+}$.¹

The main difference between these off-forward parton distributions (OFPDs) and the usual (forward) parton distribution functions is that $p' \neq p$ in Eq. (1.1), i.e. since the initial state is different from the final state in the above matrix element, OFPDs cannot be interpreted as parton densities. Although it is not yet entirely clear what intuitive physical information is contained in the off-forward distributions, it is already clear that they play a ‘dual’ role between form-factors and parton distribution functions [1,2]: when one integrates Eq. (1.1) over x one obtains the matrix element of a local operator, i.e. form factors. On the other hand, in the forward limit $\zeta = 0$ and $t = 0$, one recovers the usual parton densities, and the case $\zeta = 0$, $t \neq 0$ is related to impact parameter dependent parton distributions. In a certain sense one can

therefore interpret OFPDs as a ‘parton decomposition of form factors’, because $f_\zeta(x, t)$ gives the contribution of partons with momentum fraction x to the form factor $F(t)$.

Therefore, OFPDs might be of tremendous help in studying the light-cone wave functions of nucleons. For example, in the light-cone formalism, form factors for $q^2 < 0$ can be evaluated in the Drell-Yan-West frame where $q^+ = 0$. In this frame, the current is diagonal in Fock space and form factors can be expressed in terms of overlap integrals between light-cone wave functions of the target (see e.g. Ref. [4]). Compared to the form factors, OFPDs give a much more detailed information about light-cone wave functions since they involve one less integration.

One of the great difficulties in applying light-cone wave function based phenomenology to form factors with $q^2 > 0$ is that one can no longer go to a frame where $q^+ = 0$ and thus the current operator is not necessarily diagonal in Fock space. As a result, form factors for $q^2 > 0$ can no longer be expressed as overlap integrals involving only the target’s light-cone wave function, but one must also include contributions where the current first forms a meson which subsequently merges with the target. OFPDs provide the unique opportunity to separate these contributions, since the region $x > \zeta$ involves only contributions where the current is diagonal in Fock space, while $x < \zeta$ corresponds to the regime where the current creates a $q\bar{q}$ pair.

Despite the similarities between ordinary parton distributions and OFPDs in certain limiting cases, the physical interpretation of OFPDs in general is still less obvious. This fact makes it particularly difficult to model OFPDs since it is not clear what properties one should expect and therefore toy model studies may be very useful during such a development stage. In such a situation, where one is studying a new observable with little known properties, it is often very useful to consider solvable toy models.

For this purpose, we will use the ‘t Hooft model [5] (1+1 dimensional QCD for $N_C \rightarrow \infty$) as an exactly solvable toy model to study OFPDs in a relativistic field theory. Despite being just a toy model, QCD_{1+1} has a number of interesting features, such as confinement, in common with QCD_{3+1} . Of course, although toy models cannot predict what really happens in 3+1 dimensional QCD, they can provide useful insight on typical features that one should expect in a relativistic field theory: unlike most phenomenological models, the ‘t Hooft model

¹Note that we have chosen to follow here the conventions of Ref. [2], which differ slightly from those in Ref. [1].

describes a covariant field theory and its solutions still reflect the implicit constraints inherent to relativistic field theories, such as for example boost invariance and analyticity.

The paper is organized as follows. In Section II, form factors for the ‘t Hooft model are expressed in terms of light-cone wave functions. because of the close connection between form factors and OFPDs, these form the basis for deriving explicit expressions for OFPDs in terms of light-cone wave functions in Section III. Explicit numerical examples are presented in Section IV, and the results are summarized in Section V.

II. PARTON DISTRIBUTION FUNCTIONS AND FORM FACTORS IN TWO-DIMENSIONAL QCD

In the ‘t Hooft model [5], i.e. two dimensional QCD in the large N_C limit, mesons are described as (non-interacting) bound states of a quark-antiquark pair ($a\bar{b}$)

$$M^2\Psi(x) = \left(\frac{m_a^2}{x} + \frac{m_b^2}{1-x}\right)\Psi(x) + \bar{g}^2 \int_0^1 dy \frac{\Psi(x) - \Psi(y)}{(x-y)^2}. \quad (2.1)$$

Here x denotes the momentum fraction carried by the quark in the infinite momentum (light-front) frame. $\bar{g}^2 \equiv \frac{g^2 N_C}{2\pi}$ is kept fixed in the $N_C \rightarrow \infty$ limit. The eigenvalues M_n^2 of Eq. (2.1) form a discrete spectrum and correspond to the invariant masses of meson resonances in this model.

Since the ‘t Hooft model is formulated in a light-front framework, parton distribution, which have the physical meaning of momentum densities in the infinite momentum frame, can be easily calculated from the eigenfunctions to the ‘t Hooft equation. One thus finds for the distribution function of the quark a in state with light-cone wave function $\Psi(x)$ [6]

$$f_a(x) = |\Psi(x)|^2. \quad (2.2)$$

In Ref. [6], one can also find exact expressions for the vector form factor. For details, the reader is referred to this very useful paper. For example, for the matrix element of the $+$ component of the vector current, which couples only to the quark, between two meson states one finds

$$\langle m, p' | j^+(0) | n, p \rangle = 2P^+ F_{mn}^+(\zeta) \quad (2.3)$$

where

$$F_{mn}^+(\zeta) = \int_\zeta^1 dx \Psi_m^* \left(\frac{x-\zeta}{1-\zeta} \right) \Psi_n(x) + \bar{g}^2 \zeta^2 \int_0^1 dv \int_\zeta^1 dy \Psi_m^* \left(\frac{y-\zeta}{1-\zeta} \right) \frac{\Psi_n(\zeta v) - \Psi_n(y)}{(y-\zeta w)^2} G(v; t). \quad (2.4)$$

and

$$G(u; t) = \int_0^1 dv G(u, v; t) \\ G(u, v; t) = \sum_k \frac{\phi_k(u) \phi_k(v)}{t - \mu_k^2 + i\varepsilon} \quad (2.5)$$

is the ‘Green’s function for an $a\bar{a}$ pair, i.e. the μ_n^2 and ϕ_n in Eq. (2.5) are eigenvalues and eigenfunctions to the ‘t Hooft equation with $m_{\bar{b}} = m_a$.

Here $q = p - p'$, $\zeta = \frac{q^+}{p^+}$ and $t = q^2$. In 1+1 dimensions, there is no transverse momentum and therefore the invariant momentum transfer t is related to the ‘off-forwardness’ ζ through energy conservation, i.e.

$$M_n^2 = \frac{t}{\zeta} + \frac{M_m^2}{1-\zeta}. \quad (2.6)$$

In Section III, we will provide a interpretation of OFPDs in terms of light-cone time ordered diagrams. Since the Fock space interpretation of the various terms in form factors is very similar to the terms contributing to OFPDs, the reader is referred to Fig. 1 for this purpose.

The first term on the r.h.s. in Eq. (2.4) arises from pieces in the current operator which are diagonal in Fock space. Those terms can be directly evaluated from the overlap integral between the light-cone momentum space wave functions in the two particle sector. The second term on the r.h.s. in Eq. (2.4) corresponds to pair creation terms in the current operator, i.e. physically it corresponds to the emission of a virtual meson which is subsequently absorbed by the external current. Naively, one may not have expected that such a contribution survives the $N_C \rightarrow \infty$ limit where meson-meson vertices are suppressed by one power of $\frac{1}{\sqrt{N_C}}$. However, the vacuum to meson matrix element of the current operator scales like $\sqrt{N_C}$, which compensates for the $\frac{1}{\sqrt{N_C}}$ suppression of the triple meson vertex.

The Green’s function $G(u, v; t)$, in the second term on the r.h.s. of Eq. (2.4), describes the non-perturbative interaction of the $q\bar{q}$ pair emanating from the external current before it interacts with the target meson. Following Ref. [6], we describe this interaction by inserting a complete set of meson states. It should be emphasized that Eq. (2.4) is exact to leading order in $\frac{1}{N_C}$, i.e. it should have all properties that one would expect from a form factor in a relativistic field theory. For further details, as well as a derivation of Eq. (2.4), the reader is referred to Ref. [6].

III. OFF FORWARD DISTRIBUTION FUNCTIONS

Since Einhorn’s result for the form factors (2.4) is already expressed in terms of light-cone variables, it is a

straightforward exercise to isolate the off forward distribution functions. For $x > \zeta$, only the piece diagonal in Fock space contributes and one finds (Fig. 1a)

$$f_\zeta(x) = \Psi^* \left(\frac{x-\zeta}{1-\zeta} \right) \Psi(x) \quad (x > \zeta). \quad (3.1)$$

Note that Eq. (3.1) saturates the positivity constraint for OFPDs [7]. Eq. (3.1) reduces to the usual parton distribution [Eq. (2.2)] for $\zeta = 0$, i.e.

$$f_{\zeta=0}(x) = f(x). \quad (3.2)$$

For $x < \zeta$, the bilocal current in Eq. (1.1) creates a $q\bar{q}$ pair with momentum fraction ζ , i.e. only the off diagonal piece contributes and one finds (Fig. 1b,c)

$$f_\zeta(x) = f_\zeta^b(x) + f_\zeta^c(x) \quad (x < \zeta), \quad (3.3)$$

where

$$f_\zeta^b(x) = \bar{g}^2 \int_0^\zeta dw \int_\zeta^1 dy \Psi^* \left(\frac{y-\zeta}{1-\zeta} \right) \frac{\Psi(w)}{(y-w)^2} G \left(\frac{w}{\zeta}, \frac{x}{\zeta}; t \right) \quad (3.4)$$

and

$$f_\zeta^c(x) = -\bar{g}^2 \int_0^\zeta dw \int_\zeta^1 dy \Psi^* \left(\frac{y-\zeta}{1-\zeta} \right) \frac{\Psi(y)}{(y-w)^2} G \left(\frac{w}{\zeta}, \frac{x}{\zeta}; t \right). \quad (3.5)$$

Like Eq. (2.4), Eqs. (3.1) and (3.3) are exact to leading order in $\frac{1}{N_C}$!

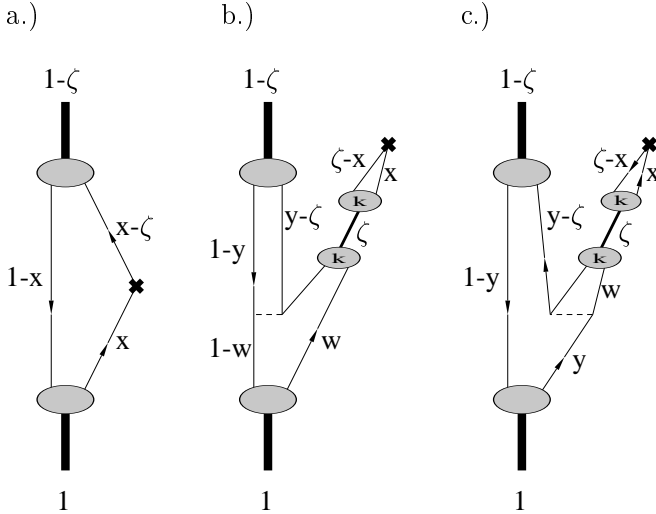


FIG. 1. Light-cone time-ordered diagrams, illustrating the physics of the various contributions to $f_\zeta(x)$. a.) Diagrams diagonal in Fock space contributing for $x > \zeta$. b.), c.) diagrams off-diagonal in Fock space, which are relevant for $x < \zeta$. All momenta are in units of p^+ . The larger gray blobs represent the (light-cone) wave function of the incoming and outgoing meson respectively. The two smaller blobs connected by a meson propagator in b.) and c.) represent the insertion of a complete set of meson states.

The physical interpretation of Eq. (3.3) is as follows: for $x < \zeta$, the current in Eq. (1.1) creates a quark (with momentum $x p^+$ and an antiquark with momentum $(\zeta - x) p^+$. This $q\bar{q}$ pair can of course interact with itself before merging with the external meson states. Similar to the expression for the form factor, this interaction among the $q\bar{q}$ pair can be described by the Green's function $G(u, v; t)$, which suggests the same physical picture of a $q\bar{q}$ pair, which forms a meson before being absorbed by the target meson.

IV. NUMERICAL RESULTS

In the numerical calculations, we used basis functions of the form

$$\chi_n(x) = x^\beta (1-x)^\beta P_n(x), \quad (4.1)$$

where β is determined from the boundary condition $\pi\beta \cot \pi\beta = 1 - \frac{m^2}{\bar{g}^2}$ and $P_n(x)$ is a polynomial of n -th order. We found that $n \leq 10$ basis function are sufficient to achieve numerical convergence (first five figures of ground state masses stable and OFPDs did not show any visible dependence on the size of the basis).

The 't Hooft equation (2.1) contains two dimensional parameters: the gauge coupling \bar{g} and the quark mass m . Of course, since the 't Hooft model is only a toy model, there is no fundamental reason to associate any parameters in the model with experimental numbers in the real world. However, since we are trying to gain an intuitive understanding in general for regimes where quark masses are very small to 'medium sized' (u,d and s quarks), we adopt the following procedure to fix the parameters: First the coupling \bar{g} is adjusted to yield the physical string tension in QCD, i.e. we demand

$$\frac{\pi}{2} \bar{g}^2 \stackrel{!}{=} 0.18 (GeV)^2, \quad (4.2)$$

yielding $\bar{g} = 340 MeV$.

The quark masses are determined by demanding that the masses of ground state $q\bar{q}$ and $q\bar{s}$ mesons coincide with the masses of π and K mesons respectively, yielding

$$\begin{aligned} m_q &= .045 \bar{g} \\ m_s &= 1.1 \bar{g}. \end{aligned} \quad (4.3)$$

Numerical results for the OFPD $f_\zeta(x)$ for $\bar{q}q$ and $\bar{s}s$ ground state mesons are shown in Figs. 2 and 4 respectively. Results for the first excited meson state are shown in Fig. 3. For the π meson in the 't Hooft model the distribution amplitude is nearly flat. As a result, the parton distribution function ($\zeta = 0$ in Fig. 2), which is just the square of the wave function in this model, is also nearly flat. For nonzero ζ , Fig. 2 exhibits several interesting features. First of all, $f_\zeta(1) = f_\zeta(\zeta) = f_\zeta(0) = 0$. Secondly, $f_\zeta(x)$ is nearly flat for $0 < \zeta < x$ as well as for $\zeta < x < 1$.

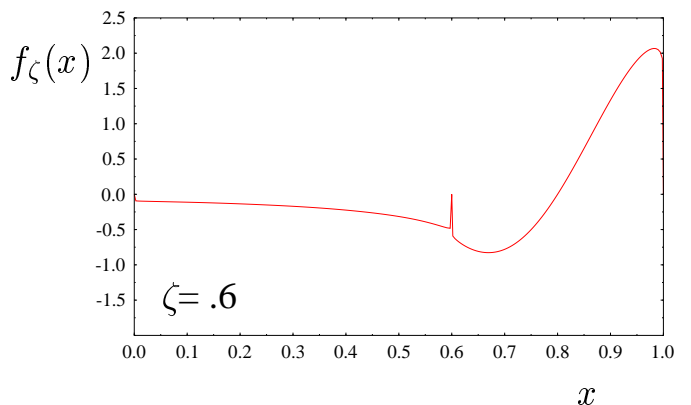
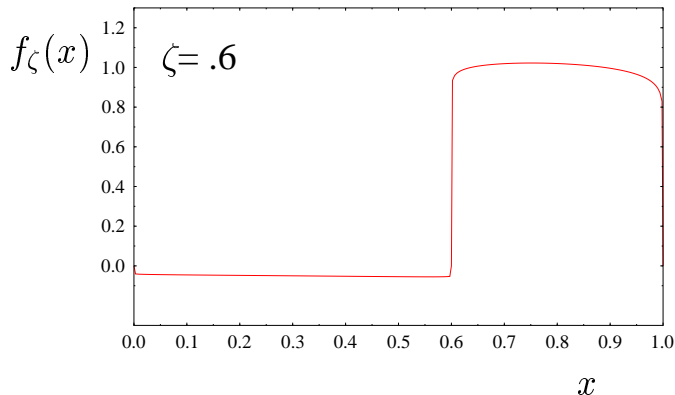
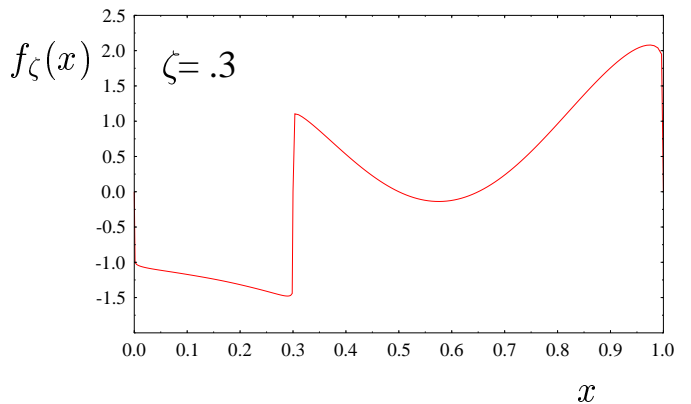
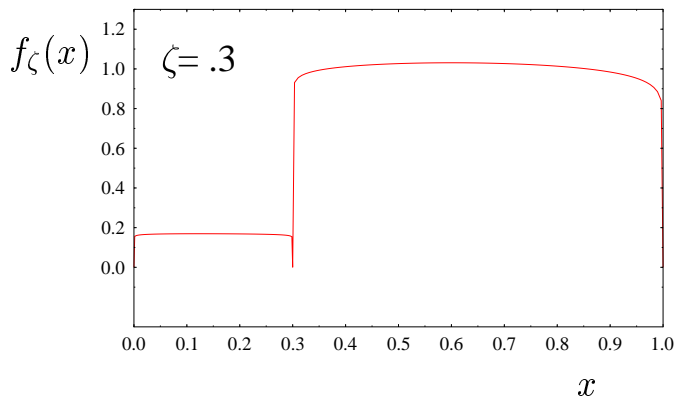
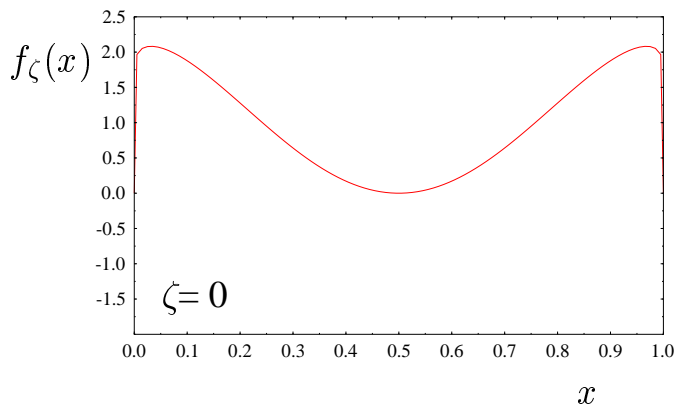
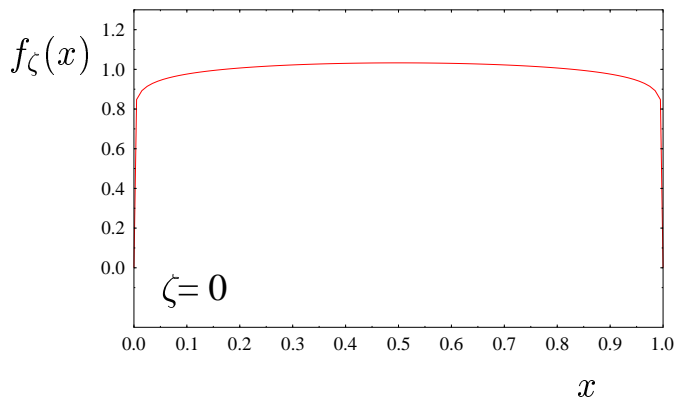


FIG. 2. Off-forward parton distributions in a $\bar{q}q$ meson with light quarks ($m_\pi = 140 MeV$) for $\zeta = 0, 0.3$ and 0.6 . The $\zeta = 0$ result is the conventional parton distribution.

In the 't Hooft model, the (target-) meson wave function is nearly flat (for small m_q) and vanishes (sharply) at the boundary. Since $f_\zeta(x)$ is just the overlap of the wave function with itself, these features are reflected in $f_\zeta(x)$ for $\zeta < x < 1$, i.e. rapid rise from zero near $x = \zeta$, nearly flat for $\zeta < x < 1$ and a rapid drop near $x = 1$. In 3+1 dimensional QCD one would also expect that $f_\zeta(1) = 0$ because parton distributions vanish at that point as well. However, since parton distributions in QCD_{3+1} do not

FIG. 3. Same as Fig. 2, but for the first orbitally excited meson state.

vanish at $x = 0$, one would not expect $f_\zeta(\zeta)$ to vanish either. The physics of the regime $0 < x < \zeta$ is entirely different. Here the x dependence comes from the x dependence of the distribution amplitude of the meson that couples to the external current. In general, that contribution is complicated since there is an infinite sum of mesons contributing. However, when t is near a pole in this sum, that particular meson will of course dominate and as a result the x dependence will be proportional to the distribution amplitude of that particular meson.

This is the case for small quark masses in QCD_{1+1} ! In two dimensions, the ‘ π ’-meson couples not only to pseudoscalar but also to vector currents and thus the ‘ π ’ does contribute to the sum over meson poles in the Green’s function relevant for vector currents (2.5).

In two dimensions, t and ζ are not independent variables, but are related to each other (and the target mass) through Eq. (2.6). Since the pion mass goes to zero for small m_q and since t is proportional to the target mass² (here also the π) we are in a situation where the π pole contribution dominates in the sum over meson poles. As a result, the x dependence for $0 < x < \zeta$ is, to a good approximation, proportional to $\Psi_\pi(\frac{x}{\zeta})$. The coefficient of proportionality depends on ζ in a complicated way [described by Eq. (3.3)]. In fact it even changes sign around $\zeta \approx 0.5$ since there are two competing terms with opposite sign.²

This close connection between OFPD for $0 < x < \zeta$ and distribution amplitudes as for example been pointed out already in Ref. [2] where it was noticed that OFPD evolve with the Brodsky-Lepage kernel for distribution amplitudes when $0 < x < \zeta$, i.e. the UV part of OFPD and distribution amplitudes are the same in this regime.

In fact, one can use this connection to measure the light-cone distribution amplitudes of mesons contributing in the sums over meson poles: the 3 + 1 dimensional generalization of Eq. (3.3) reads

$$f_\zeta(x, t) = \sum_n \frac{c_n(\zeta, t)}{t - M_n^2} \phi_n\left(\frac{x}{\zeta}\right), \quad (4.4)$$

where $\phi_n(z)$ is the n -th meson’s distribution amplitude, and $c_n(\zeta, t)$ are some coefficients characterizing the coupling of the mesons to the target. It is therefore conceivable that, by extrapolating in t to the lowest meson pole, to use OFPD in the regime $0 < x < \zeta$ as a tool to measure meson distribution amplitudes (e.g. the ρ -meson distribution amplitude, which is otherwise hard to access)

Although $f_\zeta(x)$ in QCD_{1+1} is continuous and vanishes for $x = 0, \zeta$ and 1, these properties are not very clearly visible in Fig. 2 due to the rapid rise of the ‘t Hooft wave function near the endpoints for small m_q . These features are better illustrated for heavier quarks, where the wave function rises less rapidly near the endpoints. Results for ‘strange’ quarks³ are shown in Fig. 4.

Up to trivial kinematic factors, $\int_0^1 dx f_\zeta(x)$ yields the form factor. For a ground state meson target, even for small quark masses, and for typical values of ζ , Fig. 2 shows that most of the contribution to this integral arises

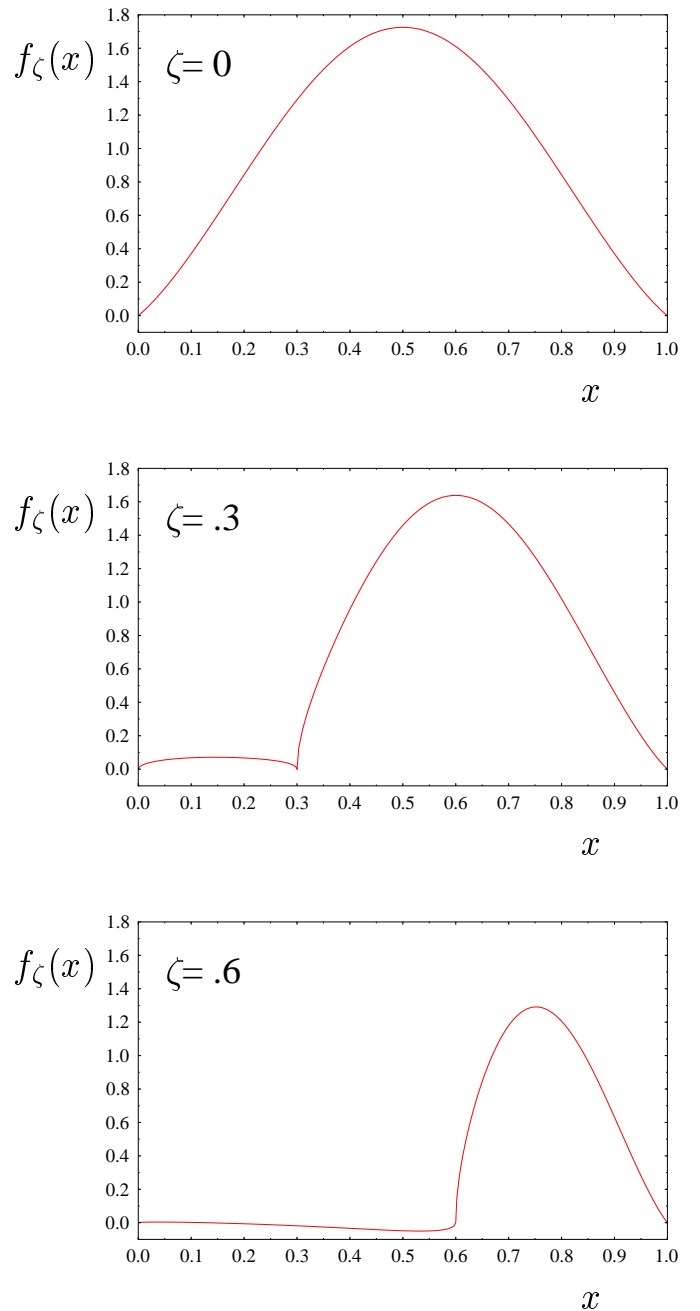


FIG. 4. Same as Fig. 2 but for a quark mass on the order of the strange quark mass.

from the region $x > \zeta$, i.e. from the terms that are diagonal in Fock space (3.1). This is a very surprising result since it should cost only very little energy to create a $q\bar{q}$ fluctuation which could then couple to the target.⁴ Nu-

²As is illustrated in Fig. 1 b. and c., the gluon from the annihilation vertex can be absorbed either by the quark or the antiquark!

³See the discussion above on how the parameters in the ‘t Hooft equation were fixed.

⁴Note that the lightest meson in QCD_{1+1} , which becomes massless as $m_q \rightarrow 0$, *does* contribute to the sum over meson poles in Eq. (3.3).

merically, there is of course a large cancellation between the two terms in Eq. (3.3). These two terms correspond to the two possibilities that the instantaneous ‘gluon’, which emerges when the antiquark from the current gets absorbed, couples to the quark or antiquark in the target meson. Because these two terms almost seem to cancel each other, it is suggestive to interpret this result in terms of a ‘screening phenomenon’, indicating a ‘small size’⁵ for the ground state meson in the chiral limit. However, beyond this very handwaving picture, we have no intuitive explanation for this surprising dynamical suppression of the off-diagonal terms in the form factor for the ground state meson in this model.

V. SUMMARY

We have presented exact expressions for off forward parton distributions in the ‘t Hooft model (QCD_{1+1} with $N_C \rightarrow \infty$). These expressions, together with their numerical evaluation, illustrate a number of features that should be generic for OFPDs in any relativistic field theory (regardless of the number of dimensions).

Comparing light-front Fock space overlap integrals for form factors (2.4) and OFPDs (3.1,3.3)⁶, it is evident that OFPDs can be interpreted as ‘parton decompositions of form factors’. In fact, the only difference between form factors and OFPDs is the fact that in OFPDs the momentum of the ‘probed quark’ is not integrated over but rather kept fixed at momentum fraction x . Of course, in the ‘t Hooft model, where the wave functions depend only on one parameter, and therefore the Fock-space diagonal contribution to the form factor involves only one integration, this means that the OFPD is given by a mere product of the wave function with itself. This illustrates that, particularly for large x , where contributions from higher Fock components should be suppressed also in 3+1 dimensions, OFPDs provide much more direct information on the light-cone wave function of the target than form factors.

While the x dependence for $x > \zeta$ depends only on properties of the target hadron’s light-cone Fock space wave functions, this is no longer the case for $x < \zeta$. In the latter regime, the x dependence is governed by the light-cone wave functions of mesons coupling to the external current. In fact, Eq. (3.3) reflects that for a fixed value of ζ , each meson contributes with some ζ -dependent coefficient times its light-cone wave function. Although it will in general be difficult to disentangle this

sum over meson poles into contributions from individual mesons, the lightest state with the appropriate quantum numbers should give a dominant contribution for small t . For example, for unpolarized OFPDs, where the γ -matrix structure of the probe is that of a vector current, one would be most sensitive to contributions from ρ and ω mesons and the success of the vector meson dominance picture for form factors suggests that there should be little contamination from higher states at small t . In the case of polarized OFPDs, where the γ -matrix structure is that of an axial vector current, the dominance of the lowest pole (the π) should be even more pronounced at small t .

In the ‘t Hooft model we found that $f_\zeta(x)$ vanishes for $x = \zeta$ and is continuous at this point. The fact that $f_\zeta(x)$ vanishes for $x \rightarrow \zeta$ from below comes from the fact that meson distribution amplitudes vanish at the end-points in this model and, at least if distribution amplitudes are close to their asymptotic form in QCD_{3+1} , one would expect the same behavior there as well. The situation is different for $x \rightarrow \zeta$ from above, since there one would expect significant contributions from higher Fock components. In fact, $f_\zeta(x)$ might even diverge at this point.

In the introduction, we pointed out that OFPDs allow to decompose form factors into their dependence on the light-cone momentum of the active parton. It is very remarkable that, despite the fact that a vector meson dominance description for the form factor works extremely well for small m_q [8], only a very small portion of the form factor for a ground state meson comes from the $x < \zeta$ region. Instead, for typical values of ζ , most of the contributions to the form factor arise from $x > \zeta$, where the current is diagonal in Fock space and the form factor can be expressed as an overlap between light-cone wave functions. It would be very interesting to study if a similar suppression of the $x < \zeta$ contributions also takes place in QCD_{3+1} .

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⁵It is not clear which definition of ‘size’ really sets the scale here, which is why we were unable to quantify this argument any further.

⁶For $\zeta > x$ the corresponding expressions in QCD_{3+1} can for example be found in Ref. [4].

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- [1] X. Ji, Phys. Rev. Lett. **78**, 610 (1997); Phys. Rev. D **52**, 3841 (1995).
 - [2] A.V. Radyushkin, Phys. Rev. D **56**, 5524 (1997).
 - [3] M. Burkardt, hep-ph/0005108.
 - [4] M. Diehl et al., Eur. Phys. J. **C8**, 409 (1999); Phys. Lett. B **460**, 204 (1999)
 - [5] G. ‘t Hooft, Nucl. Phys. B **75**, 461 (1974).
 - [6] M.B. Einhorn, Phys. Rev. D **14**, 3451 (1976).
 - [7] B. Pire, J. Soffer and O. Teryaev, Eur. Phys. J. **C8**, 103, 1999.
 - [8] R.L. Jaffe and P. Mende, Nucl. Phys. B **369**, 189 (1992).